

Homogenization Method, uncovered on 2 pages

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This note gives a brief and ‘crash’ introduction to the method of Homogenization with the use of wave equation and diffusion equation with periodic in space coefficients as instructive examples. We expose the method with the use of an approach, which appears in Vibrodynamics. The note can be interesting for people who want to use the Homogenization method immediately.

1. Introduction

Homogenization represents a mathematical method actively used in high impact interdisciplinary research. There are hundreds of related publications, even the list of recently published monographs contains about twenty titles, including Hornung (1997), Kozlov, Papanicolaou, Berdichevskii, Zhikov (1999), Donato & Cioranescu (2000), Marchenko & Khruslov (2005), Chechkin, Piatnitski, Shamaev (2007), Pavliotis & Stuart (2008), Tartar (2010), Gilbert & Panchenko (2016). This literature shows that the using of Homogenization theory often requires rather advanced mathematical techniques, such as functional analysis, etc. The aim of this note is to give a simplest possible ‘crash’ introduction to the method of Homogenization by showing how it works for the description of waves in an elastic medium with periodically changing properties. The exposition follows to systematic and justifiable procedure that was introduced in Vibrodynamics, see Vladimirov (2005), Yudovich (2006), Vladimirov (2010). The results include the deriving of equation for an averaged part of solution and an explicit formula for its oscillating part. As a result, the equation for the averaged solution appears as the wave equation with an effective coefficient of elasticity, which is a well-known and classical result. Transparency, generality, and simplicity are the main benefits of our approach.

2. Formulation of problem and the two-scale setting

We consider one-dimensional elastic medium with the density and elasticity properties periodically changing in the x -direction. The wave equation for a longitudinal displacement $a = a(x, t)$ is written in the dimensional form as

$$\boxed{\rho a_{tt} = (E a_x)_x; \quad \rho(x) = \rho(x + 2\pi l), \quad E(x) = E(x + 2\pi l)} \quad (2.1)$$

where t - time, subscripts stand for partial derivatives, $E > 0$ - Yong’s modulus, $\rho > 0$ - mass per unit length, $2\pi l = \text{const} > 0$ - period in x . For functions ρ and E we take

$$\rho = \rho_0 + \tilde{\rho} > 0, \quad E = E_0 + \tilde{E} > 0 \quad (2.2)$$

where ρ_0, E_0 are constants, and $\tilde{\rho}, \tilde{E}$ are $2\pi l$ -periodic in x functions with zero average. For example, $\rho = \rho_0 + \rho_1 \cos(x/l), \quad E = E_0 + E_1 \cos(x/l)$ with constants ρ_1, E_1 . We

introduce the dimensionless variables and parameters (marked with ‘dags’) as follows:

$$x = lx^\dagger, \quad t = Tt^\dagger, \quad E = E_0E^\dagger, \quad \rho = \rho_0\rho^\dagger; \quad T \equiv \lambda\sqrt{\rho_0/E_0}$$

The dimensionless form of (2.1) is (the ‘dags’ are dropped):

$$(Ea_x)_x = \varepsilon^2 \rho a_{tt}; \quad \varepsilon \equiv l/\lambda, \quad \rho(x) = \rho(x + 2\pi), \quad E(x) = E(x + 2\pi) \quad (2.3)$$

We accept that ε is a small parameter $\varepsilon \equiv l/\lambda \ll 1$, which means that we consider the large-scale solutions (of characteristic scale $\lambda \gg l$).

To solve (2.3) we introduce *two mutually dependent* spatial variables ξ and r

$$\xi = x, \quad r = \varepsilon x$$

and the form of the unknown function which depends on both variables

$$a(x, t) = b(\xi, r, t)$$

With the use of the chain rule $\partial/\partial x = \partial/\partial \xi + \varepsilon \partial/\partial r$ the equation (2.3) takes the form

$$(Eb_\xi)_\xi + \varepsilon[(Eb_\xi)_r + (Eb_r)_\xi] + \varepsilon^2[(Eb_r)_r - \rho b_{tt}] = 0 \quad (2.4)$$

The key ‘multi-scale’ or ‘two-scale’ assumption is:

$$\boxed{\text{We consider } \xi \text{ and } r \text{ as two mutually independent variables}}$$

Hence we are looking for solutions $b(\xi, r, t)$ of the PDE (2.4) with three independent variables t, ξ and r . We express the solutions to (2.4) as regular series

$$b = b_0 + \varepsilon b_1 + \varepsilon^2 b_2 + \dots, \quad b_i = b_i(\xi, r, t) = O(1), \quad i = 1, 2, 3, \dots \quad (2.5)$$

For performing of asymptotic procedure we assume that all dimensionless function $b(\xi, r, t)$, $b_i(\xi, r, t)$ (say, denoted as f) has the following properties:

- $f \sim O(1)$ and all its ξ -, r -, and t -derivatives are also $O(1)$;
- f is 2π -periodic in ξ , *i.e.* $f(\xi, s, \tau) = f(\xi + 2\pi, r, t)$;
- f has an average given by

$$\langle f \rangle \equiv \frac{1}{2\pi} \int_{\xi_0}^{\xi_0 + 2\pi} f(\xi, r, t) d\xi \equiv \bar{f}(r, t) \quad \forall \xi_0 = \text{const}; \quad (2.6)$$

- f can be split into the average and purely oscillating parts $f(\xi, r, t) = \bar{f}(r, t) + \tilde{f}(\xi, r, t)$, where *tilde-functions* (or purely oscillating functions) are such that $\langle \tilde{f} \rangle = 0$ and the *bar-functions* (or averaged functions) $\langle f \rangle \equiv \bar{f}$ are ξ -independent.

3. Asymptotic Procedure

Let us describe the procedure of solving (2.4), (2.5) for the first three successive approximations:

The equation of order ε^0 is

$$(E\tilde{b}_{0\xi})_\xi = 0 \quad \text{or} \quad E\tilde{b}_{0\xi} = \bar{F}_1(r, t) \quad (3.1)$$

where $\bar{F}_1(r, t)$ is an arbitrary function, which appears after integration over ξ . It is convenient to rewrite (3.1) as

$$\tilde{b}_{0\xi} = M\bar{F}_1(r, t); \quad \text{where} \quad M \equiv 1/E, \quad E > 0$$

Its bar-part is $\overline{M} \overline{F}_1 = 0$, which immediately yields $\overline{F}_1 = 0$. Then (3.1) gives

$$\boxed{\widetilde{b}_{0\xi} = 0 \quad \text{or} \quad b_0 = \overline{b}_0(r, t)} \quad (3.2)$$

with unknown function $\overline{b}_0(r, t)$. Surprisingly, the solution in main approximation is not oscillating.

The equation of order ε^1 is

$$(E\widetilde{b}_{1\xi})_\xi + (Eb_{0\xi})_r + (Eb_{0r})_\xi = 0$$

The use of (3.2) simplifies it to the form

$$(E\widetilde{b}_{1\xi} + E\overline{b}_{0r})_\xi = 0$$

which after integration gives

$$\widetilde{b}_{1\xi} + \overline{b}_{0r} = M\overline{F}_2(r, t) \quad (3.3)$$

with an arbitrary function $\overline{F}_2(r, t)$. Its bar-part yields $\overline{F}_2(r, t) = \overline{b}_{0r}/\overline{M}$, then the tilde-part of (3.3) leads to

$$\boxed{\widetilde{b}_{1\xi} = \overline{b}_{0r} \widetilde{M}/\overline{M}} \quad (3.4)$$

This formula is the only result obtainable from the equation of the first approximation.

The equation of order ε^2 is

$$(E\widetilde{b}_{2\xi})_\xi + (Eb_{1r})_\xi + (E\widetilde{b}_{1\xi})_r + (Eb_{0r})_r - \rho b_{0tt} = 0$$

its bar-part is

$$\langle \widetilde{E}\widetilde{b}_{1\xi} \rangle_r + (\overline{E}\overline{b}_{0r})_r - \overline{\rho}\overline{b}_{0tt} = 0$$

which jointly with (3.4) leads to:

$$\overline{\rho}\overline{b}_{0tt} = [(\overline{E} + \langle \widetilde{E}\widetilde{M} \rangle / \overline{M}) \overline{b}_{0r}]_r \quad (3.5)$$

The substitution $\widetilde{E} = E - \overline{E}$, $\widetilde{M} = M - \overline{M}$ yields $\overline{E} + \langle \widetilde{E}\widetilde{M} \rangle / \overline{M} = 1/\overline{M}$ and the final form of (3.5) is

$$\boxed{\overline{\rho}\overline{b}_{0tt} = (E_{\text{eff}}\overline{b}_{0r})_r, \quad \text{where} \quad E_{\text{eff}} = \langle E^{-1} \rangle^{-1}} \quad (3.6)$$

where the calculation of average $\langle \cdot \rangle$ represents the integration over ξ (2.6).

Hence, we have derived the wave equation for the solution in the main approximation, with an effective Young's module $E_{\text{eff}} = \text{const}$. One can notice, that we haven't used any assumption for the smallness of oscillation parts in the given functions E and ρ (2.2). Similar results represent the main and typical outcomes of all Homogenization approaches. One can also notice, that in addition to the equation (3.6) for the averaged part \overline{b}_0 of the solution we have also obtained the explicit formula for its oscillating part b_1 (3.4).

4. Discussion

In all the above consideration we have exploited only the x -derivatives in the equation, and have left the time derivative in its original form. Hence, it is possible to consider any other equation (PDE) containing the same x -derivatives (and different time-derivatives) applying the same two-scale procedure as above, to have the same outcome for the

‘spatial’ part of the equation. The best example to demonstrate this statement is to consider the diffusion equation

$$a_t = (Ka_x)_x, \quad K(x) = K_0 + \tilde{K} \quad (4.1)$$

where $a = a(x, t)$ – is a passive admixture or temperature, K_0 is constant, $K = K(x) > 0$ – the diffusion coefficient which is assumed to be a periodic function $K(x) = K(x + 2\pi)$. The procedure is identical to the given above and the expression for the effective diffusion coefficient is similar to E_{eff} in (3.6). Hence, the averaged diffusion equation is:

$$\bar{a}_{0t} = (K_{\text{eff}}\bar{a}_{0r})_r, \quad K_{\text{eff}} = \langle K^{-1} \rangle^{-1} \quad (4.2)$$

Interesting examples showing that the exposed method is highly efficient and useful can be easily continued.

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